JOURNAL OF APPROXIMATION THEORY 65, 151–175 (1991)

On Polynomials Orthogonal with Respect to Certain Sobolev Inner Products

A. ISERLES

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, England

P. E. KOCH AND S. P. NØRSETT

Division of Mathematical Sciences, Norwegian Institute of Technology, Trondheim, Norway

AND

J. M. SANZ-SERNA

Department of Applied Mathematics and Computation, University of Valladolid, Valladolid, Spain

Communicated by Alphonse P. Magnus

Received February 6, 1990; revised June 29, 1990

We are concerned with polynomials $\{p_n^{(\lambda)}\}$ that are orthogonal with respect to the Sobolev inner product

$$\langle f, g \rangle_\lambda = \int fg \, d\varphi + \lambda \int f'g' \, d\psi,$$

where λ is a non-negative constant. We show that if the Borel measures $d\varphi$ and $d\psi$ obey a specific condition then the $p_n^{(\lambda)}$'s can be expanded in the polynomials orthogonal with respect to $d\varphi$ in such a manner that, subject to correct normalization, the expansion coefficients, except for the last, are independent of n and are themselves orthogonal polynomials in λ. We explore several examples and demonstrate how our theory can be used for efficient evaluation of Sobolev–Fourier coefficients. © 1991 Academic Press, Inc.

1. INTRODUCTION

Orthogonality in Sobolev spaces has attracted considerable attention in recent decades [1, 3–6, 8, 10, 17–20]. In the present paper we propose to approach that oft-discussed problem from a different point of view.

0021-9045/91 $3.00

640/65/2-3

Throughout, the term "Borel measure" is used to refer to positive Borel measures φ on the real line, satisfying the conditions

$$\left|\int_{-\infty}^{\infty} x^k \, d\varphi(x)\right| < \infty, \qquad k = 0, 1, 2, \ldots \tag{1.1}$$

(finite moments) and

$$\int_{-\infty}^{\infty} p(x) \, d\varphi(x) > 0 \tag{1.2}$$

for each polynomial p that is non-negative for all real x and not identically zero (positive definiteness).

Thus, if $d\varphi$ and $d\psi$ are two Borel measures and λ is a positive constant, the Sobolev space $W_2^1[(-\infty, \infty), d\varphi, d\psi]$ with the inner product

$$\langle f, g \rangle_\lambda := \int_{-\infty}^{\infty} f(x) \, g(x) \, d\varphi(x) + \lambda \int_{-\infty}^{\infty} f'(x) \, g'(x) \, d\psi(x) \tag{1.3}$$

contains the space of all polynomials and it makes sense to study polynomials that are orthogonal with respect to $\langle \cdot, \cdot \rangle_\lambda$. Specifically, we say that the nth degree polynomial $p_n^{(\lambda)}$, $p_n^{(\lambda)} \not\equiv 0$, is *Sobolev-orthogonal* if

$$\langle x^k, p_n^{(\lambda)} \rangle_\lambda = 0, \qquad k = 0, 1, \ldots, n-1. \tag{1.4}$$

Note that, of course, the parameter λ can be absorbed into $d\psi$. This, however, will defeat one of the purposes of our investigation: to examine the dependence of $p_n^{(\lambda)}$ upon λ. Note further that the condition (1.2) is equivalent to the requirement that the support of $d\varphi$ should contain an infinite number of points. In other words, we do not allow discrete measures of the form

$$\int_{-\infty}^{\infty} f(x) \, d\theta(x) = \sum_{k=0}^{n} \theta_k f(x_k).$$

Much is already known about Sobolev-orthogonal polynomials. Some of the familiar properties of the "standard" orthogonal polynomials can be translated intact into the more general framework, with obvious amendments: for example, among all nth degree polynomials with a fixed coefficient of x^n, $p_n^{(\lambda)}$ minimizes the norm that is induced by $\langle \cdot, \cdot \rangle_\lambda$ [10]. Expansion of functions in $\{p_n^{(\lambda)}\}_{n=0}^{\infty}$ with respect to the inner product (1.3) is also a straightforward extension of the classical theory—cf. [1, 4, 18]—

and it has attracted recent attention [6] because of its relevance to the analysis of spectral methods for partial differential equations. However, most properties fail or need be paraphrased to a large extent: three-term recurrence relations are, of course, lost, as is the Christoffel–Darboux formula. More importantly, zeros of $p_n^{(\lambda)}$ need not belong to the support of the underlying measures [1].

Two sets of measures have been investigated in some detail:

$$\varphi(x) = \psi(x) = \begin{cases} -1: & x < -1; \\ x: & -1 \leqslant x \leqslant 1; \\ 1: & 1 < x \end{cases}$$

(the Legendre case) and

$$\varphi(x) = \psi(x) = \begin{cases} -1: & x < 0; \\ -e^{-x}: & 0 \leqslant x \end{cases}$$

(the Laguerre case). This led to the derivation of explicit forms, recurrence relations, and localisation of zeros [1, 4, 8].

Another problem that has recently received some attention is that of Sobolev orthogonality with atomic measures, inclusive of the case of $d\psi$ being supported on a finite set [17, 20]. Although, strictly speaking, it ceases in that case to be a Borel measure, $\langle \cdot, \cdot \rangle_\lambda$ is, nonetheless, a positive inner product, as long as $d\varphi$ is a genuine measure.

In the present paper we study the expansion of $p_n^{(\lambda)}$ in the basis spanned by $p_0, p_1, ..., p_n$, where the p_n's are orthogonal (in the usual sense) with respect to the inner product defined by the first Borel measure, $d\varphi$. In other words, $p_n \equiv D_n p_n^{(0)}$, where $D_n \not\equiv 0$ is a constant. Naturally, an expansion of the form

$$p_n^{(\lambda)}(x) = \sum_{k=1}^{n} r_k^{(n)}(\lambda)\, p_k(x), \qquad n = 1, 2, ...$$

(it follows from (1.4) that $r_0^{(n)} \equiv 0$ for $n > 0$), is always well defined. However, it turns out that, subject to an extra condition being imposed on the measures $\{d\varphi, d\psi\}$ and under correct normalization, the coefficient $r_k^{(n)}$ depends only on its subscript for all $k = 0, 1, ..., n-1$. In other words, subject to the p_k's being correctly normalized, there exist $\alpha_1(\lambda), \alpha_2(\lambda), ...$ such that

$$p_n^{(\lambda)}(x) = \sum_{k=1}^{n-1} \alpha_k(\lambda)\, p_k(x) + r_n^{(n)}(\lambda)\, p_n(x)$$

for every $n = 1, 2, \ldots$. This leads *inter alia* to a recurrence relation of the form

$$p_{n+1}^{(\lambda)}(x) - p_n^{(\lambda)}(x) = \sigma_n(\lambda)(p_{n+1}(x) - p_n(x)),$$

where σ_n can be written explicitly in terms of the α_k's. The last identity can be employed to evaluate efficiently expansions of W_2^1 functions in Sobolev-orthogonal polynomials. Section 2 of the present paper is devoted to the formal derivation of the aforementioned results, whereas in Section 3 we study specific examples of $\{d\varphi, d\psi\}$ that obey the required condition and in Section 6 we present the fast algorithm for the evaluation of Fourier–Sobolev coefficients.

Section 4 is concerned with an extension of our framework, whereby $p_n^{(\lambda)}$ can be expanded in p_m's with m and n of the same parity and the coefficients of this expansion, except for the last, are independent of n.

The coefficients α_k (note that we can remove the superscript) turn out to be themselves orthogonal polynomials, and we devote Section 5 to the determination of underlying measures.

2. Coherent Pairs

Let $d\varphi$ and $d\psi$ be two Borel measures and $\lambda \geqslant 0$ a given constant. We recall the definitions (1.3) and (1.4) of a Sobolev inner product $\langle \cdot, \cdot \rangle_\lambda$ and a Sobolev-orthogonal polynomial $p_n^{(\lambda)}$. Moreover, we let p_n and q_n denote orthogonal polynomials with respect to $d\varphi$ and $d\psi$, respectively:

$$\int_{-\infty}^{\infty} p_m(x)\, p_n(x)\, d\varphi(x) = \delta_{m,n} c_n,$$

$$\int_{-\infty}^{\infty} q_m(x)\, q_n(x)\, d\psi(x) = \delta_{m,n} e_n, \qquad m, n = 0, 1, \ldots,$$

where $\delta_{m,n}$ is the delta of Kronecker. The p_n's are normalized so that $p_n^{(0)} \equiv p_n$. We define numbers $d_{m,n} = d_{n,m}$ by

$$d_{m,n} = \int_{-\infty}^{\infty} p'_m(x)\, p'_n(x)\, d\psi(x), \qquad m, n = 0, 1, \ldots.$$

Obviously, since $\{p_0, p_1, \ldots, p_n\}$ span the linear space of nth degree polynomials, there exist $r_0^{(n)}, r_1^{(n)}, \ldots, r_n^{(n)}$ such that

$$p_n^{(\lambda)}(x) = \sum_{k=0}^{n} r_k^{(n)} p_k(x). \tag{2.1}$$

Of course, the r_k's are functions of λ. In fact, it will be shown below that, under appropriate normalization, the r_k's are polynomials in λ. We note that, since (1.4) with $k=0$ implies that

$$\int_{-\infty}^{\infty} p_n^{(\lambda)}(x)\,d\varphi(x)=0, \qquad n\geqslant 1,$$

necessarily $r_0^{(n)}=0$ for all $n\geqslant 1$ and the sum extends from $k=1$.

According to the definition (1.4), $p_n^{(\lambda)}$ annihilates $p_0, ..., p_{n-1}$ in the Sobolev inner product. We utilize the expansion (2.1) to obtain $n-1$ homogeneous linear equations in the n unknowns $r_1^{(n)}, r_2^{(n)}, ..., r_n^{(n)}$:

$$\begin{aligned}\langle p_n^{(\lambda)}, p_m\rangle_\lambda &\\ &= \sum_{k=1}^{n} r_k^{(n)}(\lambda)\left\{\int_{-\infty}^{\infty} p_k(x)\,p_m(x)\,d\varphi(x)+\lambda\int_{-\infty}^{\infty} p_k'(x)\,p_m'(x)\,d\psi(x)\right\}\\ &= c_m r_m^{(n)}(\lambda)+\lambda\sum_{k=1}^{n} d_{k,m} r_k^{(n)}(\lambda)=0, \qquad m=1, 2, ..., n-1.\end{aligned}$$

We complement this system with the normalizing equation

$$c_n r_n^{(n)}(\lambda)+\lambda\sum_{k=1}^{n} d_{k,n} r_k^{(n)}(\lambda)=\omega(\lambda),$$

where ω will be assigned a specific value in the sequel. Reverting to a vector form, we have the system

$$(\mathbf{C}+\lambda\mathbf{D})\,\mathbf{r}^{(n)}=\omega\mathbf{e}_n, \tag{2.2}$$

where

$$\mathbf{C}:=\operatorname{diag}\{c_1, c_2, ..., c_n\}, \qquad \mathbf{D}:=(d_{k,l})_{k,l=1,...,n},$$

$\mathbf{r}^{(n)}:=[r_1^{(n)}, r_2^{(n)}, ..., r_n^{(n)}]^T$, and $\mathbf{e}_n$ is the nth unit vector. The formal solution of (2.2) by Cramer's rule is

$$\mathbf{r}^{(n)}=\omega(\mathbf{C}+\lambda\mathbf{D})^{-1}\mathbf{e}_n=\omega\frac{\operatorname{adj}(\mathbf{C}+\lambda\mathbf{D})}{\det(\mathbf{C}+\lambda\mathbf{D})}\mathbf{e}_n, \tag{2.3}$$

where adj **B** is the "adjoint" matrix of **B**. This motivates our choice of normalization,

$$\omega:=\frac{\det(\mathbf{C}+\lambda\mathbf{D})}{\prod_{k=1}^{n-1} c_k}.$$

Operating on (2.3) yields at once

$$p_n^{(\lambda)} = [r_1^{(n)}, r_2^{(n)}, \ldots, r_n^{(n)}] \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

$$= \frac{1}{\prod_{k=1}^{n-1} c_k}$$

$$\times \det \begin{bmatrix} c_1 + \lambda d_{1,1} & \lambda d_{1,2} & \cdots & \lambda d_{1,n-1} & p_1 \\ \lambda d_{2,1} & c_2 + \lambda d_{2,2} & \cdots & \lambda d_{2,n-1} & p_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda d_{n-1,1} & \lambda d_{n-1,2} & \cdots & c_{n-1} + \lambda d_{n-1,n-1} & p_{n-1} \\ \lambda d_{n,1} & \lambda d_{n,2} & \cdots & \lambda d_{n,n-1} & p_n \end{bmatrix}. \quad (2.4)$$

Note that our choice of ω implies that, indeed, $p_n^{(0)} \equiv p_n$.

The last formula is, as such, neither surprising nor very interesting for arbitrary Borel measures. Things, however, change when we confine our attention to a subset of measures that possess an important feature: We say that the pair $\{d\varphi, d\psi\}$ of Borel measures is *coherent* if there exist non-zero constants $C_1, C_2, \ldots$ such that, for all $k, m = 1, 2, \ldots$, $C_k C_m d_{k,m}$ is a function of $\min\{k, m\}$ only. Without much danger of confusion, we write

$$d_{k,m} = \frac{d_{\min\{k,m\}}}{C_k C_m}.$$

We renormalize the underlying polynomials

$$C_m p_m \mapsto p_m, \qquad C_m p_m^{(\lambda)} \mapsto p_m^{(\lambda)}, \qquad m = 0, 1, \ldots.$$

Hence $d_{k,m} \mapsto d_{\min\{k,m\}}$ and $C_m^2 c_m \mapsto c_m$. This produces the expression

$$p_n^{(\lambda)} = \frac{1}{\prod_{k=1}^{n-1} c_k} \det \begin{bmatrix} c_1 + \lambda d_1 & \lambda d_1 & \cdots & \lambda d_1 & p_1 \\ \lambda d_1 & c_2 + \lambda d_2 & \cdots & \lambda d_2 & p_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda d_1 & \lambda d_2 & \cdots & c_{n-1} + \lambda d_{n-1} & p_{n-1} \\ \lambda d_1 & \lambda d_2 & \cdots & \lambda d_{n-1} & p_n \end{bmatrix}.$$

Next we subtract the bottom row from the remaining row to obtain

$$p_n^{(\lambda)}=\frac{1}{\prod_{k=1}^{n-1} c_k} \times \det \begin{bmatrix} c_1 & \lambda(d_1-d_2) & \lambda(d_1-d_3) & \cdots & \lambda(d_1-d_{n-1}) & p_1-p_2 \\ 0 & c_2 & \lambda(d_2-d_3) & \cdots & \lambda(d_2-d_{n-1}) & p_2-p_n \\ 0 & 0 & c_3 & \cdots & \lambda(d_3-d_{n-1}) & p_3-p_n \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & c_{n-1} & p_{n-1}-p_n \\ \lambda d_1 & \lambda d_2 & \lambda d_3 & \cdots & \lambda d_{n-1} & p_n \end{bmatrix}. \tag{2.5}$$

Except for the bottom row, we have an upper triangular matrix. To evaluate its determinant in a closed form, we seek Gaussian elimination coefficients to induce zeros into the bottom row: These are functions $\alpha_1(\lambda), \alpha_2(\lambda), ..., \alpha_{n-1}(\lambda)$ such that

$$\lambda d_m - \lambda \sum_{k=1}^{m-1} (d_m - d_k)\, \alpha_k + c_m \alpha_m = 0, \qquad m = 1, 2, ..., n-1. \tag{2.6}$$

Note that the α_m's do not depend on n.

LEMMA 1. *The solution of* (2.6) *obeys the three-term recurrence relation*

$$\begin{aligned} &c_{m+1}(d_m - d_{m-1})\, \alpha_{m+1} \\ &\quad = (c_m(d_{m+1} - d_{m-1}) + \lambda(d_{m+1} - d_m)(d_m - d_{m-1}))\, \alpha_m \\ &\qquad - c_{m-1}(d_{m+1} - d_m)\, \alpha_{m-1}. \end{aligned} \tag{2.7}$$

Proof. We set

$$\begin{aligned} \Phi := c_{m+1}(d_m - d_{m-1})\, \alpha_{m+1} - \{c_m(d_{m+1} - d_{m-1}) \\ + \lambda(d_{m+1} - d_m)(d_m - d_{m-1})\}\, \alpha_m - c_{m-1}(d_{m+1} - d_m)\, \alpha_{m-1}. \end{aligned} \tag{2.8}$$

Rearranging (2.6) yields

$$c_m \alpha_m = \lambda \left(\sum_{k=1}^{m-1} (d_m - d_k)\, \alpha_k - d_m \right).$$

We replace $c_l \alpha_l$ for $l \in \{m-1, m, m+1\}$ in (2.8) by the above expression to obtain

$$\begin{aligned}\Phi = \lambda \Bigg\{ &(d_m - d_{m-1}) \left(\sum_{k=1}^{m} (d_{m+1} - d_k)\, \alpha_k - d_{m+1} \right) \\ &- (d_{m+1} - d_{m-1}) \left(\sum_{k=1}^{m-1} (d_m - d_k)\, \alpha_k - d_m \right) \\ &- (d_{m+1} - d_m) \left(\sum_{k=1}^{m-2} (d_{m-1} - d_k)\, \alpha_k - d_{m-1} \right) \\ &- (d_{m+1} - d_m)(d_m - d_{m-1})\, \alpha_m \Bigg\} = 0.\end{aligned}$$

The proof follows. ▮

Note that, in order that (2.7) defines α_{m+1}, it is necessary that $d_m - d_{m-1} \neq 0$. It can be easily seen that this is, in fact, the case: if $d_m = d_{m-1}$ then the mth and $(m-1)$st rows of the matrix $\mathbf{D}$ coincide and $\mathbf{D}$ is singular. This is impossible, because $\mathbf{D}$ is a Gram matrix of linearly independent polynomials. An alternative proof is implicit in the statement of the second corollary to Theorem 2.

The three-term recurrence relation (2.7) is accompanied by the initial conditions

$$\alpha_1(\lambda) = -\frac{d_1}{c_1}\lambda; \tag{2.9}$$

$$\alpha_2(\lambda) = -\frac{d_2}{c_2}\lambda - \frac{d_1(d_2 - d_1)}{c_1 c_2}\lambda^2. \tag{2.10}$$

Once the α_m's have been determined by (2.7), (2.9)-(2.10), we evaluate the determinant in (2.5) by forming the product of the diagonal terms, except that we must replace p_n by the outcome of the elimination procedure in the (n, n)th entry of the matrix, namely $p_n + \sum_{k=1}^{n-1} \alpha_k (p_k - p_n)$. The result is

$$p_n^{(\lambda)}(x) = \sum_{m=1}^{n-1} \alpha_m(\lambda)\, p_m(x) + \left(1 - \sum_{m=1}^{n-1} \alpha_m(\lambda)\right) p_n(x). \tag{2.11}$$

Comparing with (2.1), it follows at once that we can identify $r_m^{(n)}$ with α_m for $m = 1, 2, ..., n-1$. In particular, it is a consequence of our analysis that $r_m^{(n)}$ is independent of n for $m = 1, ..., n-1$. Before we formulate our results in a theorem, we need to take care of the coefficient of p_n.

The identity (2.6) implies that

$$\lambda \sum_{k=1}^{m-1} d_k \alpha_k = -c_m \alpha_m - \lambda d_m + \lambda d_m \sum_{k=1}^{m-1} \alpha_k. \tag{2.12}$$

We shift the index by one in the last equation and add $\lambda d_{m-1}\alpha_{m-1}$ to both sides to obtain

$$\lambda \sum_{k=1}^{m-1} d_k \alpha_k = -c_{m-1}\alpha_{m-1} - \lambda d_{m-1} + \lambda d_{m-1} \sum_{k=1}^{m-1} \alpha_k. \tag{2.13}$$

Next, we solve Eqs. (2.12)–(2.13) for $\sum_{k=1}^{m-1} \alpha_k$ and $\sum_{k=1}^{m-1} d_k \alpha_k$. This yields

$$\sum_{k=1}^{m-1} \alpha_k = 1 + \frac{c_m \alpha_m - c_{m-1}\alpha_{m-1}}{\lambda(d_m - d_{m-1})};$$

$$\sum_{k=1}^{m-1} d_k \alpha_k = \frac{c_m d_{m-1} \alpha_m - c_{m-1} d_m \alpha_{m-1}}{\lambda(d_m - d_{m-1})}.$$

Substitution into (2.11) finally produces the expansion of Sobolev-orthogonal polynomials:

THEOREM 2. *If $\{d\varphi, d\psi\}$ form a coherent pair then*

$$p_n^{(\lambda)}(x) = \sum_{k=1}^{n-1} \alpha_k(\lambda)\, p_k(x) - \frac{c_n \alpha_n(\lambda) - c_{n-1}\alpha_{n-1}(\lambda)}{\lambda(d_n - d_{n-1})}\, p_n(x), \tag{2.14}$$

where $\alpha_1, \alpha_2, ..., \alpha_{n-1}$ obey the three-term recurrence relation (2.7). Consequently, the coefficients $r_k^{(n)}$, $k = 1, ..., n-1$, depend only on their subscript. ∎

COROLLARY. *Subject to coherence, it is true that*

$$p_{n+1}^{(\lambda)}(x) - p_n^{(\lambda)}(x) = -\frac{c_{n+1}\alpha_{n+1}(\lambda) - c_n \alpha_n(\lambda)}{\lambda(d_{n+1} - d_n)}\,(p_{n+1}(x) - p_n(x)). \tag{2.15}$$

Proof. We subtract (2.14) from a corresponding expression for $p_{n+1}^{(\lambda)}$. This produces

$$p_{n+1}^{(\lambda)}(x) - p_n^{(\lambda)}(x) = -\frac{c_{n+1}\alpha_{n+1}(\lambda) - c_n\alpha_n(\lambda)}{\lambda(d_{n+1} - d_n)}\, p_{n+1}(x) + \alpha_n(\lambda)\, p_n(x) + \frac{c_n\alpha_n(\lambda) - c_{n-1}\alpha_{n-1}(\lambda)}{\lambda(d_n - d_{n-1})}\, p_n(x). \tag{2.16}$$

It follows from (2.6) that

$$\frac{c_{n+1}\alpha_{n+1}(\lambda) - c_n\alpha_n(\lambda)}{d_{n+1} - d_n} = \lambda\alpha_n(\lambda) + \frac{c_n\alpha_n(\lambda) - c_{n-1}\alpha_{n-1}(\lambda)}{d_n - d_{n-1}}$$

and substitution in (2.16) furnishes the required expression. ∎

The definition of coherence may seem at first sight rather strange and difficult to verify. Fortunately, we are able to show that it is equivalent to a condition that is far less technical.

THEOREM 3. *The pair $\{d\varphi, d\psi\}$ is coherent if and only if there exist non-zero constants $C_1, C_2, \ldots$ such that*

$$q_n(x) = C_{n+1}\, p'_{n+1}(x) - C_n\, p'_n(x), \qquad n = 1, 2, \ldots. \tag{2.17}$$

Proof. Let us first assume that $\{d\varphi, d\psi\}$ is coherent. Hence there exist non-zero $C_1, C_2, \ldots$ such that

$$\int_{-\infty}^{\infty} p'_m(x)\, p'_k(x)\, d\psi(x) = \frac{d_{\min\{m,k\}}}{C_m C_k}, \qquad k, m = 1, 2, \ldots$$

and it follows at once that

$$\int_{-\infty}^{\infty} p'_m(x)(C_{n+1}\, p'_{n+1}(x) - C_n\, p'_n(x))\, d\psi(x) = 0, \qquad m = 1, 2, \ldots, n.$$

But $\{p'_1, p'_2, \ldots, p'_n\}$ spans all $(n-1)$th degree polynomials, hence, (2.17) is true.

The opposite statement follows just as readily, be reversing our argument: assuming that (2.17) holds, we have

$$0 = \int_{-\infty}^{\infty} p_m(x)\, q_n(x)\, d\psi(x) = \int_{-\infty}^{\infty} p'_m(x)(C_{n+1}\, p'_{n+1}(x) - C_n\, p'_n(x))\, d\psi(x)$$

for all $m = 1, 2, \ldots, n$, therefore,

$$d_{m,n+1} = \frac{C_n}{C_{n+1}}\, d_{m,n}, \qquad m = 1, 2, \ldots, n.$$

It follows that $C_n C_m d_{n,m}$ is independent of n for all $m \leqslant n$ and depends only on $m = \min\{m, n\}$—precisely the definition of coherence. ∎

COROLLARY. *Subject to coherence and assuming, without loss of generality, that $C_n \equiv 1$, it is true that*

$$d_{n+1} - d_n = e_n > 0, \qquad n = 0, 1, \ldots. \tag{2.18}$$

In particular, $d_{n+1} - d_n$ never vanishes.

Proof. Subtracting

$$d_n = \int_{-\infty}^{\infty} p'_n(x)\, p'_{n+1}(x)\, d\psi(x)$$

from

$$d_{n+1} = \int_{-\infty}^{\infty} p'^{\,2}_{n+1}(x)\, d\psi(x)$$

results in

$$\begin{aligned} d_{n+1} - d_n &= \int_{-\infty}^{\infty} p'_{n+1}(x)(p'_{n+1}(x) - p'_n(x))\, d\psi(x) \\ &= \int_{-\infty}^{\infty} p'_{n+1}(x)\, q_n(x)\, d\psi(x) = \int_{-\infty}^{\infty} q_n^2(x)\, d\psi(x) = e_n. \end{aligned}$$

Note that we have exploited the identity

$$\int_{-\infty}^{\infty} p'_n(x)\, q_n(x)\, d\psi(x) = 0$$

in the derivation of the last expression. ∎

COROLLARY. *Subject to coherence and assuming, without loss of generality, that $C_n \equiv 1$, it is true that*

$$\frac{d}{dx}\left(p^{(\lambda)}_{n+1}(x) - p^{(\lambda)}_n(x)\right) = -\frac{c_{n+1}\alpha_{n-1}(\lambda) - c_n\alpha_n(\lambda)}{\lambda(d_{n-1} - d_n)}\, q_n(x). \tag{2.19}$$

Proof. A straightforward juxtaposition of Theorems 2 and 3. ∎

Before we conclude this section it is worthwhile to note that the three-term recurrence relation (2.7) can be rewritten in a simplified form involving $e_n = d_{n+1} - d_n$,

$$\begin{aligned} &c_{m+1} e_{m-1} \alpha_{m+1}(\lambda) \\ &\quad = (c_m(e_m + e_{m-1}) + \lambda e_{m-1} e_m)\, \alpha_m(\lambda) - c_{m-1} e_m \alpha_{m-1}(\lambda). \end{aligned} \tag{2.20}$$

3. Coherent Pairs and Classical Orthogonal Polynomials

Given two Borel measures $d\varphi$ and $d\psi$, we say that $d\psi$ is a *companion* of $d\varphi$ if $\{d\varphi, d\psi\}$ form a coherent pair.

In the present section we assume that $d\varphi$ is a measure that produces one of the classical orthogonal polynomials—Jacobi, Laguerre, or Hermite—and seek its companions. According to a theorem of Hahn [11], classical orthogonal polynomials are precisely all the orthogonal polynomials whose derivatives are also orthogonal with respect to some Borel measure $d\upsilon$. Let us denote the monic orthogonal polynomials with respect to $d\upsilon$ by $\pi_0, \pi_1, \ldots$. Thus,

$$\pi_n(x) = \frac{1}{n+1} p'_{n+1}(x), \qquad n = 0, 1, \ldots,$$

where we stipulate, without loss of generality, that the p_n's are monic. The main tool of our analysis is the identity (2.17), which expresses the q_n's, subject to coherence, in terms of the π_n's.

Suppose that the π_n's obey the recurrence relation

$$\pi_{n+1}(x) = (x - \alpha_n)\, \pi_n(x) - \beta_n \pi_{n-1}(x). \tag{3.1}$$

We let

$$\begin{aligned} q_0(x) &:\equiv 1, \\ q_n(x) &:= \pi_n(x) - \sigma_n \pi_{n-1}(x), \qquad n = 1, 2, \ldots \end{aligned} \tag{3.2}$$

(hence the q_n's are quasi-orthogonal with respect to $d\upsilon$ [7]) and seek real constants $\sigma_1, \sigma_2, \ldots \neq 0$ such that $\{q_n\}_{n=0}^{\infty}$ is an orthogonal polynomial system. According to the Favard theorem [7], this is the case if and only if the q_n's obey a three-term recurrence relation, which we write as

$$q_{n+1}(x) = (x - \gamma_n)\, q_n(x) - \delta_n q_{n-1}(x). \tag{3.3}$$

Substitution of (3.2) into (3.3) yields

$$\begin{aligned} \pi_{n+1}(x) &= (x + \sigma_{n+1} - \gamma_n)\, \pi_n(x) \\ &\quad - (\sigma_n x - \sigma_n \gamma_n + \delta_n)\, \pi_{n-1}(x) + \sigma_{n-1} \delta_n \pi_{n-2}(x). \end{aligned}$$

We now exploit (3.1) to replace π_{n+1} and π_{n-2} in the last formula. This gives

$$\begin{aligned} \left(\alpha_n + \sigma_{n+1} - \gamma_n - \frac{\sigma_{n-1}\delta_n}{\beta_{n-1}}\right) \pi_n(x) &= \left(\left(\delta_n - \beta_n - \sigma_n \gamma_n + \frac{\sigma_{n-1}\delta_n \alpha_{n-1}}{\beta_{n-1}}\right)\right. \\ &\quad \left. + \left(\sigma_n - \frac{\sigma_{n-1}\delta_n}{\beta_{n-1}}\right) x\right) \pi_{n-1}(x). \end{aligned} \tag{3.4}$$

Let us examine (3.4). Since π_{n-1} and π_n cannot share zeros, both sides of the equation must necessarily vanish. Thus, we obtain

$$\alpha_n + \sigma_{n+1} - \gamma_n - \frac{\sigma_{n-1}\delta_n}{\beta_{n-1}} = 0, \tag{3.5}$$

$$\beta_n + \sigma_n \gamma_n - \delta_n - \frac{\sigma_{n-1}\delta_n \alpha_{n-1}}{\beta_{n-1}} = 0, \tag{3.6}$$

$$\sigma_n - \frac{\sigma_{n-1}\delta_n}{\beta_{n-1}} = 0. \tag{3.7}$$

The identity (3.7) yields

$$\delta_n = \frac{\sigma_n}{\sigma_{n-1}} \beta_{n-1}$$

(recall that the σ_n's do not vanish!) and substitution in (3.5)–(3.6) results in

$$\alpha_n + \sigma_{n+1} - \gamma_n - \sigma_n = 0, \tag{3.8}$$

$$\frac{\beta_{n-1}}{\sigma_{n-1}} - \frac{\beta_n}{\sigma_n} + \alpha_{n-1} = \gamma_n. \tag{3.9}$$

Substitution of (3.9) into (3.8) gives

$$\sigma_{n+1} + \alpha_n + \frac{\beta_n}{\sigma_n} = \sigma_n + \alpha_{n-1} + \frac{\beta_{n-1}}{\sigma_{n-1}}.$$

Consequently, the quantity $\sigma_{n+1} + \alpha_n + \beta_n/\sigma_n$ is independent of n and there exists a real number ξ such that

$$\sigma_{n+1} + \alpha_n + \frac{\beta_n}{\sigma_n} = \xi, \qquad n = 0, 1, ... \tag{3.10}$$

We now set

$$A_0 :\equiv 1,$$

$$A_n := \sigma_1 \sigma_2 \cdots \sigma_n, \qquad n = 1, 2, ...$$

Thus, $\sigma_n = A_n / A_{n-1}$ and substitution into (3.10) affirms that each A_n is a monic nth degree polynomial in ξ that obeys the recurrence relation

$$A_{n+1}(\xi) = (\xi - \alpha_n) A_n(\xi) - \beta_n A_{n-1}(\xi). \tag{3.11}$$

Note that the recurrence relations (3.1) and (3.11) are identical, although, of course, they are subject to different initial conditions. The general solution of (3.11) can be expressed as $\{a\pi_n(x) + (1-a)\,\pi_n^{(1)}(x)\}_{n=0}^{\infty}$, where the $\pi_n^{(1)}$'s are the numerator polynomials corresponding to the measure $d\upsilon$ [7].

It now follows from (3.3) that, for all $n = 1, 2, \ldots$,

$$q_n(x) = \frac{1}{A_{n-1}(\xi)}\,(A_{n-1}(\xi)\,\pi_n(x) - A_n(\xi)\,\pi_{n-1}(x)). \tag{3.12}$$

Polynomials of this form were already investigated in [9, 12, 15, 16, 21, 22], mainly because of their connection with Christoffel weights of certain quadrature formulae. In particular, Maroni [21] proved that, subject to ξ being outside the (open) essential support of $d\upsilon$ and to $d\upsilon$ corresponding to one of the classical orthogonal polynomials, they are orthogonal with respect to the measure

$$(1-c)\,\delta(x-\xi)\,dx + c\,\frac{d\upsilon(x)}{|x-\xi|}, \tag{3.13}$$

where $0 < c \leqslant 1$.

In the case of a Jacobi measure,

$$d\varphi(x) = (1-x)^{\alpha}\,(1+x)^{\beta}\,dx, \qquad x \in (-1, 1),$$

where $\alpha, \beta > -1$, it is well known that

$$d\upsilon(x) = (1-x)^{\alpha+1}\,(1+x)^{1+\beta}\,dx, \qquad x \in (-1, 1)$$

[23]. Thus, letting $\xi = -1$ and $c = 1$ in (3.13), we obtain

$$d\psi(x) = (1-x)^{1+\alpha}\,(1-x)^{\beta}\,dx, \qquad x \in (-1, 1)$$

as a particular example of a companion of $d\varphi$.

Likewise, the Laguerre measure

$$d\varphi(x) = x^{\alpha} e^{-x}\,dx, \qquad x > 0,$$

where $\alpha > -1$, yields

$$d\upsilon(x) = x^{\alpha+1} e^{-x}\,dx, \qquad x > 0$$

[23]. Thus, an example of a measure that results in a coherent pair follows when we choose $\xi = 0$ and $c = 1$ in (3.13): in that case $d\psi(x) \equiv d\varphi(x)$.

Finally, it is easy to see that if

$$d\varphi(x) = e^{-x^2}\,dx, \qquad x \in (-\infty, \infty),$$

the Hermite measure, there exists no companion, since the support of $d\varphi$ (and hence of $d\upsilon$) is the whole real line and we cannot chose a real number ξ in (3.13).

4. Symmetrically Coherent Pairs

If both measures $d\varphi$ and $d\psi$ are symmetric (i.e., invariant under the transformation $x \mapsto -x$) then $p_n^{(\lambda)}$, p_n, and q_n are of the same parity as n. Therefore, expanding $p_n^{(\lambda)}$ in p_m's, only terms that share the parity of n are present. Consequently, expansion coefficients depend on parity and the pair $\{d\varphi, d\psi\}$ cannot be coherent. (Note the lack of symmetry of the companions $(1-x)^{\alpha+1}(1+x)^{\alpha}\,dx$, $(1-x)^{\alpha}(1+x)^{\alpha+1}\,dx$ found above for the symmetric Jacobi measure $(1-x)^{\alpha}(1+x)^{\alpha}\,dx$, $\alpha > -1$.) Fortunately, a relatively minor generalization of coherence caters for this situation.

Given symmetric $d\varphi$ and $d\psi$, we say that $\{d\varphi, d\psi\}$ is *symmetrically coherent* if

$$d_{k,m} = \begin{cases} C_k C_m d_{\min\{k,m\}}: & k \text{ and } m \text{ are of the same parity;} \\ 0: & \text{otherwise.} \end{cases}$$

Moreover, a measure $d\psi$ is termed a *symmetric companion* of $d\varphi$ if the pair $\{d\varphi, d\psi\}$ is symmetrically coherent. All the results of Section 2 translate to this framework, with obvious amendments:

THEOREM 4. (a) *The pair $\{d\varphi, d\psi\}$ of symmetric measures is symmetrically coherent if and only if there exist non-zero constants $C_1, C_2, \ldots$ such that*

$$q_n(x) = C_{n+1} p'_{n+1}(x) - C_{n-1} p'_{n-1}(x).$$

(b) *Subject to symmetric coherence and assuming that, without loss of generality, $C_n \equiv 1$, it is true that*

$$p_n^{(\lambda)}(x) = \sum_{k=1}^{[n/2]-1} \alpha_{n-2k}(\lambda)\, p_{n-2k}(x) - \frac{c_n \alpha_n(\lambda) - c_{n-2}\alpha_{n-2}(\lambda)}{\lambda(d_n - d_{n-2})}\, p_n(x),$$

where the α_k's obey the three-term recurrence relation

$$c_{m+2} e_{m-2} \alpha_{m+2}(\lambda) = (c_m(e_m + e_{m-2}) + \lambda e_{m-2} e_m)\, \alpha_m(\lambda) - c_{m-2} e_m \alpha_{m-2}(\lambda).$$

(c) *Subject to the same conditions, it is true that*

$$p_{n+1}^{(\lambda)}(x) - p_{n-1}^{(\lambda)}(x) = -\frac{c_{n+1}\alpha_{n+1}(\lambda) - c_{n-1}\alpha_{n-1}(\lambda)}{\lambda(d_{n+1} - d_{n-1})}\, (p_{n+1}(x) - p_{n-1}(x))$$

and

$$\frac{d}{dx}\left(p_{n+1}^{(\lambda)}(x)-p_{n-1}^{(\lambda)}(x)\right)=-\frac{c_{n+1}\alpha_{n+1}(\lambda)-c_{n-1}\alpha_{n-1}(\lambda)}{\lambda(d_{n+1}-d_{n-1})}\,q_n(x).$$

We need symmetric measures for symmetric coherence and our first example is the *Hermite* measure $d\varphi(x)=e^{-x^2}\,dx$, $-\infty<x<\infty$, that already has been debated in Section 3 in the context of coherent pairs. Thus, we have

$$H_{n+1}(x)=2xH_n(x)-2nH_{n-1}(x)$$

and the polynomials q_n, being themselves orthogonal with respect to a symmetric measure, obey a three-term recurrence relation of the form

$$q_{n+1}(x)=a_n x q_n(x)-f_n q_{n-1}(x). \tag{4.1}$$

Since $q_n=C_{n+1}H'_{n+1}-C_{n-1}H'_{n-1}$ and $H'_n=2nH_{n-1}$, substitution into (4.1) yields

$$\begin{aligned}(n+2)\,C_{n+2}H_{n+1}-nC_nH_{n-1}&=a_nx((n+1)\,C_{n+1}H_n-(n-1)\,C_{n-1}H_{n-2})\\&\quad-f_n(nC_nH_{n-1}-(n-2)\,C_{n-2}H_{n-3}).\end{aligned}$$

Replacing H_{n+1} with $2xH_n-2nH_{n-1}$ and H_{n-3} with $(xH_{n-2}-\frac{1}{2}H_{n-1})/(n-2)$ leads to

$$\begin{aligned}&x(2(n+2)\,C_{n+2}-(n+1)\,a_nC_{n+1})\,H_n\\&\quad+(-nC_n+nf_nC_n+\tfrac{1}{2}f_nC_{n-2}-2n(n+1)\,C_{n+2})\,H_{n-1}\\&\quad+x((n-1)\,a_nC_{n-1}-f_nC_{n-2})\,H_{n-2}=0.\end{aligned}$$

Next, we substitute $(xH_{n-1}-\frac{1}{2}H_n)/(n-1)$ instead of H_{n-2}. This yields

$$A_n^*xH_n(x)+(B_n^*+C_n^*x^2)\,H_{n-1}(x)=0,$$

where

$$A_n^*=2(n+2)\,C_{n+2}-(n+1)\,a_nC_{n+1}-\frac{1}{2}a_nC_{n-1}+\frac{1}{2}\frac{1}{n-1}f_nC_{n-2};$$

$$B_n^*=-nC_n+nf_nC_n+\tfrac{1}{2}f_nC_{n-2}-2n(n+2)\,C_{n+2};$$

$$C_n^*=a_nC_{n-1}-\frac{1}{n-1}f_nC_{n-2}.$$

As in Section 3, it follows that $A_n^* = B_n^* = C_n^* = 0$, since H_n and H_{n-1} cannot share zeros. We obtain

$$a_n = \frac{n}{n-1} \frac{C_{n-2}}{C_{n-1}} \frac{2(n+2)\, C_{n+2} + C_n}{nC_n + \frac{1}{2} C_{n-2}};$$

$$f_n = n \frac{2(n+2)\, C_{n+2} + C_n}{nC_n + \frac{1}{2} C_{n-2}}, \qquad n = 2, 3, ...,$$

where

$$\rho_n := (n+1) \frac{C_{n+1}}{C_{n-1}}$$

obeys the non-linear recurrence

$$(n-1)(2\rho_{n-1} + 1)\, \rho_{n+1} = \rho_n. \tag{4.2}$$

It is easy to see that any initial conditions $\rho_1, \rho_2 > 0$ produce in (4.2) a positive solution sequence. In that case $a_n, f_n > 0$ and the measure that generates $\{q_n\}$ is positive definite. Thus, we have an example of a measure that has an infinite number of symmetric companions (and, as we have demonstrated in Section 3, no companions, whether symmetric or not).

Our next example of a symmetric measure with a symmetric companion is the *Gegenbauer* measure $d\varphi(x) = (1 - x^2)^{\nu - 1/2}$, $x \in (-1, 1)$, where $\nu > 0$. Thus, $p_n = C_n^\nu$, the Gegenbauer polynomials [23]. Special cases include $\nu = \frac{1}{2}$, the *Legendre* measure, and $\nu = 1$, the *Chebyshev* measure of the second kind. We do not try to find all the symmetric companions—a single example will suffice. Since

$$\frac{d}{dx} \{C_{n+1}^\nu(x) - C_{n-1}^\nu(x)\} = 2(n+\nu)\, C_n^\nu(x)$$

[23], it follows at once that the Gegenbauer measure is a symmetric companion of itself.

This construction fails when $\nu = 0$, since Gegenbauer polynomials are not defined. This is an important special case, since it corresponds to the *Chebyshev* measure of the first kind. Fortunately, this measure is a companion of itself for $p_n := T_n$, $C_n := 1/n$, as can be seen at once from

$$\frac{d}{dx} \left\{ \frac{T_{n+1}}{n+1} - \frac{T_{n-1}}{n-1} \right\} = U_n - U_{n-2} = 2T_n.$$

Note that the framework of Section 3 can be extended to symmetrically coherent pairs, thereby characterizing all the symmetric companions of Gegenbauer and Hermite measures.

5. Orthogonality Relations of the Expansion Coefficients

We have seen that, $\{d\varphi, d\psi\}$ being a coherent pair, the expansion coefficients in (2.1), except for the last, obey the recurrence relation (2.20). In the present section we investigate some implications of this relation. Note that we replace λ with x, to emphasize that it is now the main variable, rather than a parameter.

THEOREM 5. *Let*

$$R_n(x) := \frac{\alpha_{n+1}(x)}{\alpha_1(x)} = -\frac{c_1}{d_1}\frac{\alpha_{n+1}(x)}{x}, \qquad n = 0, 1, \ldots.$$

Then the set $\{R_n\}_{n=0}^{\infty}$ *is orthogonal with respect to some Borel measure.*

Proof. We have from (2.9)–(2.10) and from (2.18) that

$$R_0(x) \equiv 1, \qquad R_1(x) = \frac{c_1 d_2}{c_2 d_1} + \frac{e_1}{c_2} x. \tag{5.1}$$

Moreover, the identity (2.20) is "translated" into

$$\begin{aligned} &c_{n+2} e_n R_{n+1}(x) \\ &\quad = (e_n e_{n+1} x + c_{n+1}(e_{n+1} + e_n)) R_n(x) - c_n e_{n+1} R_{n-1}(x). \end{aligned} \tag{5.2}$$

The coefficients c_m and e_m are positive for all $m = 0, 1, \ldots$. We now invoke the Favard theorem [7] to deduce that there exists a Borel measure $d\chi$ such that

$$\int_{-\infty}^{\infty} R_m(x) R_n(x)\, d\chi(x) = 0, \qquad m \neq n. \quad \blacksquare$$

The general connection between the measures $d\varphi$, $d\psi$, and $d\chi$ has not been clarified as yet. However, several special cases are amenable to analysis. It is convenient, first, to rewrite (5.1) (5.2) in terms of monic polynomials. These are $\hat{R}_n$ such that

$$\begin{aligned} &\hat{R}_0(x) \equiv 1, \qquad \hat{R}_1(x) = c_1\left(\frac{1}{e_0} + \frac{1}{e_1}\right) + x, \\ &\hat{R}_{n+1}(x) = \left\{x + c_{n+1}\left(\frac{1}{e_n} + \frac{1}{e_{n+1}}\right)\right\} \hat{R}_n(x) - \frac{c_n c_{n-1}}{e_n^2} \hat{R}_{n-1}(x); \end{aligned} \tag{5.3}$$

this can be readily verified, e.g., by using formulae from [7].

We have proved in Section 3 that the *Laguerre* measure is a companion of itself. We have $p_n = q_n = L_n^{(\alpha)}$, $\alpha > -1$, and

$$c_n = e_n = \frac{\Gamma(n+\alpha+1)}{n!}, \qquad n = 0, 1, \ldots$$

[23]. Writing $R_n^*(x) := R_n(2x-2)$, we act on (5.2) to obtain

$$(n+1)(n+2+\alpha)\, R_{n+1}^*(x) = (n+2)(2(n+1)\,x+\alpha)\, R_n^*(x) - (n+1)(n+2)\, R_{n-1}^*(x).$$

They can be converted to their monic version:

$$\hat{R}_{n+1}^*(x) = \left(x + \frac{\alpha}{2(n+1)}\right) \hat{R}_n^*(x) - \frac{n+\alpha-1}{4(n+1)}\, \hat{R}_{n-1}^*(x). \tag{5.4}$$

We compare (5.4) with the recurrence formula for the monic *Pollaczek* polynomials

$$\hat{P}_{n+1}^{\lambda}(x) = \left(x + \frac{b}{n+\lambda+a}\right) \hat{P}_n^{\lambda}(x) - \frac{n(n+2\lambda-1)}{4(n+\lambda+a-1)(n+\lambda+a)}\, \hat{P}_{n-1}^{\lambda}(x),$$

where $a, b, \lambda \in \mathscr{R}$ and $a+\lambda > 0$ [7]. It readily follows that the formulae coincide for the choice $a = -\alpha/2$, $b = \alpha/2$, and $\lambda = 1 + \alpha/2$. Consequently,

$$R_n(x) = \frac{(n+1)!}{(2+\alpha)_n}\, P_n^{1+\alpha/2}\left(1 + \frac{1}{2}x;\ -\frac{\alpha}{2}, \frac{\alpha}{2}\right).$$

Orthogonality properties of Pollaczek polynomials are known when $a \geqslant |b|$ [7]. This corresponds to the case $\alpha \in (-1, 0]$ and we have

$$d\chi(2x-2) = (1-x^2)^{(1/2)(1+\alpha)} \exp\left(\alpha\left(\cos^{-1} x - \frac{\pi}{2}\right)\left(\frac{1-x}{1+x}\right)^{1/2}\right)$$
$$\times \left|\Gamma\left(1 + \frac{\alpha}{2}\left(1 + i\left(\frac{1-x}{1+x}\right)^{1/2}\right)\right)\right|^2 dx, \qquad x \in (-1, 1).$$

The support of $d\chi$ is the interval $(-4, 0)$. If $\alpha = 0$ then $d\chi$ reduces to a shifted *Chebyshev* measure of the second kind. Note that the underlying measure has been identified for other values of a and b in [2].

The next object of our attention is the *Jacobi* measure. We proved in Section 3 that a coherent choice is

$$p_n = (-1)^{n-1} \frac{(\alpha+\beta+2)_{n-1}}{(\alpha+2)_{n-1}}\, P_n^{(\alpha,\beta)},$$

$$q_n = \frac{(-1)^n}{2} \frac{(\alpha+\beta+2)_n}{(\alpha+2)_n}\, (\alpha+\beta+2n+2)\, P_n^{(\alpha+1,\beta)}.$$

We now exploit the identity

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta (P_n^{(\alpha,\beta)}(x))^2\, dx$$

$$= \frac{2^{\alpha+\beta+1}\Gamma(1+\alpha)\,\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} \frac{(1+\alpha)_n\,(1+\beta)_n}{n!\,(1+\alpha+\beta)_n\,(1+\alpha+\beta+2n)}.$$

Setting

$$C^{(\alpha,\beta)} := \frac{2^{\alpha+\beta+1}\Gamma(1+\alpha)\,\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)},$$

we obtain

$$c_n = C^{(\alpha,\beta)} \left(\frac{1+\alpha}{1+\alpha+\beta}\right)^2 \frac{(1+\beta)_n\,(1+\alpha+\beta)_n}{n!\,(1+\alpha)_n\,(1+\alpha+\beta+2n)},$$

$$e_n = \frac{C^{(\alpha+1,\beta)}}{4} \frac{(1+\beta)_n\,(2+\alpha+\beta)_n}{n!\,(2+\alpha)_n}$$

and (5.3) gives

$$\hat{R}_{n+1}(x) = (x+a_n)\,\hat{R}_n(x) - b_n \hat{R}_{n-1}(x), \tag{5.5}$$

where

$$a_n = \frac{(3+\alpha+\beta+2n+\sqrt{(1+\alpha)^2-\beta^2})(3+\alpha+\beta+2n-\sqrt{(1+\alpha)^2-\beta^2})}{(1+n)(2+\alpha+\beta+n)(3+\alpha+\beta+2n)},$$

$$b_n = 4\frac{(1+\alpha+n)(1+\beta+n)}{(1+n)(1+\alpha+\beta+n)(1+\alpha+\beta+2n)(3+\alpha+\beta+2n)}.$$

Some combinations of α and β yield simplified recurrences (5.5). For example, letting $\beta = 0$ produces

$$\hat{R}_{n+1}(x) = \left(x + \frac{2}{n+\nu+1}\right)\hat{R}_n(x) - \frac{1}{(n+\nu)(n+\nu+1)}\hat{R}_{n-1}(x),$$

where $\nu := \frac{1}{2}(1+\alpha)$. The underlying measure $d\chi$ has not been identified.

Analysis of symmetrically coherent measures is similar to Theorem 5, except that now we have two different three-term recurrences, one for "even" polynomials E_n (i.e., those that appear in the expansion of $p_{2m}^{(\lambda)}$) and one for "odd" polynomials O_n. Our last example concerns the *Gegenbauer* measures. Since

$$\int_{-1}^{1} (1-x^2)^{\nu-1/2}\,[C_n^\nu(x)]^2\, dx = \frac{\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu)} \frac{(2\nu)_n}{n!\,(n+\nu)}$$

[23], simple calculation leads to the monic recurrences

$$\hat{E}_{n+1} = \left\{x + \frac{2\nu^2 + (4n+5)\nu + (2n+1)(2n+3)}{8(n+1)(n+\nu+1)(2n+\nu+1)(2n+\nu+3)}\right\} \hat{E}_n - \frac{(2n+1)(2n+2\nu+1)}{64(n+1)(n+\nu)(2n+\nu)(2n+\nu+1)^2(2n+\nu+2)} \hat{E}_{n-1}$$

and

$$\hat{O}_{n+1} = \left\{x + \frac{2\nu^2 + (4n+3)\nu + 4n(n+1)}{2(2n+1)(2n+\nu)(2n+\nu+2)(2n+2\nu+1)}\right\} \hat{O}_n - \frac{n(n+\nu)}{4(2n+1)(2n+\nu-1)(2n+\nu)^2(2n+\nu+1)(2n+2\nu-1)} \hat{O}_{n-1}.$$

They remain true for the *Chebyshev* measure of the first kind, i.e., when $\nu = 0$.

A very special case is the *Legendre* measure, which is obtained for $\nu = \frac{1}{2}$. In that case we have

$$\hat{E}_{n+1} = \left\{x + \frac{2}{(4n+3)(4n+7)}\right\} \hat{E}_n - \frac{1}{(4n+1)(4n+3)^2(4n+5)} \hat{E}_{n-1}; \tag{5.6}$$

$$\hat{O}_{n+1} = \left\{x + \frac{2}{(4n+1)(4n+5)}\right\} \hat{O}_n - \frac{1}{(4n-1)^2(4n+1)^2(4n+3)} \hat{O}_{n-1}. \tag{5.7}$$

To identify $d\chi^{(E)}$ and $d\chi^{(O)}$, the "even" and "odd" measures in the Legendre case, we recall for future reference that the modified *Lommel* measure has a discrete spectrum, with jumps of $1/j^2_{\nu-1,k}$ at $\pm 1/j_{\nu-1,k}$, $k = 1, 2, ...,$ where $j_{\kappa,k}$ is the kth zero of the Bessel function J_κ and $\nu > 1$. The underlying monic orthogonal polynomials posssess the three-term recurrence relation

$$\hat{h}_{n+1,\nu}(x) = x\hat{h}_{n,\nu}(x) - \frac{1}{4(n+\nu-1)(n+\nu)} \hat{h}_{n-1,\nu}(x) \tag{5.8}$$

[7].

Let $d\mu$ be a symmetric Borel measure that generates monic orthogonal polynomials $\{s_n\}$ with the three-term recurrence relation

$$s_{n+1}(x) = xs_n(x) - \lambda_n s_{n-1}(x). \tag{5.9}$$

Since s_n retains the parity of n, $t_n(x) := s_{2n+1}(\sqrt{x})/\sqrt{x}$ is a polynomial and it is well known that it is orthogonal with respect to the measure $\sqrt{x}\, d\mu(\sqrt{x})$, supported by $x \in (0, \infty)$ [7]. It is quite easy to prove that the t_n's obey the recurrence relation

$$t_{n+1}(x) = (x - \lambda_{2n+1} - \lambda_{2n+2})\, t_n(x) - \lambda_{2n}\lambda_{2n+1} t_{n-1}(x). \qquad (5.10)$$

We map $x \mapsto -x$ in (5.6). It is now straightforward to verify that (5.6) and (5.10) coincide for the choice

$$\lambda_n = \frac{1}{4(n+\nu-1)(n+\nu)}, \qquad \nu = \frac{1}{2}.$$

Thus, exploiting the connection between (5.9) and (5.10), we deduce that $d\chi^{(E)}$ is a "one-sided" modified Lommel measure. Likewise, $\nu = \frac{3}{2}$ recovers the recurrence relation for $\hat{O}_n$. But

$$J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos z, \qquad J_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z,$$

and it transpires that $d\chi^{(E)}$ is an atomic measure with jumps of $1/k^4$ at $-1/(\pi^2 k^2)$, $k = 1, 2, \ldots$, whereas $d\chi^{(O)}$ is an atomic measure with jumps of $1/(k+\frac{1}{2})^4$ at $-1/(\pi^2(k+\frac{1}{2})^2)$, $k = 1, 2, \ldots$.

6. Evaluation of Expansion Coefficients

Orthogonal polynomials can be evaluated very fast and robustly by using their three-term recurrence relations. This is of major importance in approximating the coefficients of expansions (generalized Fourier coefficients) by quadrature. We are denied the comfort of a three-term recurrence relation in the case of Sobolev-orthogonal polynomials. However, if $\{d\varphi, d\psi\}$ form a coherent or a symmetrically coherent pair, calculation of Sobolev-orthogonal polynomials and of Sobolev-Fourier coefficients can be accomplished efficiently. In the present section we present an algorithm for that purpose. Another technique, as well as a discussion of few examples and approximation-theoretical aspects, appears in [14].

Any $f \in W_2^1[(-\infty, \infty), d\varphi, d\psi]$ can be expanded in Sobolev-orthogonal polynomials,

$$f \sim \sum_{n=0}^{\infty} \frac{\hat{f}_n(\lambda)}{\hat{p}_n^{(\lambda)}}\, p_n^{(\lambda)},$$

where

$$\hat{f}_n(\lambda) = \langle f, p_n^{(\lambda)} \rangle_\lambda;$$
$$\hat{p}_n^{(\lambda)} = \langle p_n^{(\lambda)}, p_n^{(\lambda)} \rangle_\lambda.$$

We assume that $\{d\varphi, d\psi\}$ is coherent (the case of symmetric coherence is similar) and adopt the terminology of Section 2. Furthermore, we stipulate that the polynomials were normalized so that $C_m \equiv 1$. We define

$$\sigma_n(\lambda) := -\frac{c_{n+1}\alpha_{n+1}(\lambda) - c_n\alpha_n(\lambda)}{\lambda(d_{n+1} - d_n)}.$$

Thus, it follows from Theorem 2 and its corollary and from the second corollary to Theorem 3 that

$$p_{n+1}^{(\lambda)} - p_n^{(\lambda)} = \sigma_n(p_{n+1} - p_n); \tag{6.1}$$

$$p_{n+1}^{(\lambda)\prime} - p_n^{(\lambda)\prime} = \sigma_n q_n. \tag{6.2}$$

This are the key formulae that enable us to evaluate Fourier coefficients efficiently.

Let

$$(g_1, g_2)_1 := \int_{-\infty}^{\infty} g_1(x)\, g_2(x)\, d\varphi(x), \qquad (h_1, h_2)_2 := \int_{-\infty}^{\infty} h_1(x)\, h_2(x)\, d\psi(x),$$

where g_l and h_l belong to appropriate Hilbert spaces. It follows from the definition (1.3) of the Sobolev inner product that

$$\hat{f}_n = (f, p_n^{(\lambda)})_1 + \lambda(f', p_n^{(\lambda)\prime})_2;$$
$$\hat{p}^{(\lambda)} = (p_n^{(\lambda)}, p_n^{(\lambda)})_1 + \lambda(p_n^{(\lambda)\prime}, p_n^{(\lambda)\prime})_2.$$

Multiplication of identity (6.1) by f and integration yield

$$(f, p_{n+1}^{(\lambda)})_1 = (f, p_n^{(\lambda)})_1 + \sigma_n\{(f, p_{n+1})_1 - (f, p_n)_1\} \tag{6.3}$$

and (6.2) similarily leads to

$$(f', p_{n+1}^{(\lambda)\prime})_2 = (f', p_n^{(\lambda)\prime})_2 + \sigma_n(f', q_n)_2. \tag{6.4}$$

Equations (6.3) and (6.4) combine into the recurrence

$$\hat{f}_{n+1}(\lambda) = \hat{f}_n(\lambda) + \sigma_n(\lambda)\{(f, p_{n+1})_1 - (f, p_n)_1 + \lambda(f', q_n)_2\}. \tag{6.5}$$

Thus, to evaluate the $\hat{f}_n$'s it is enough to calculate first the standard generalized Fourier coefficients $\{(f, p_n)_1\}$ and $\{(f', q_n)_2\}$ and then use the recursion (6.5)—there is absolutely no need whatsoever to form Sobolev-orthogonal polynomials $p_n^{(\lambda)}$ explicitly!

To evaluate σ_n with ease we either use its definition and the recursion (2.20) or combine that recursion with the identity

$$\sigma_n = \sigma_{n-1} - \alpha_n,$$

that can be easily derived from the theory in Section 2, and the initial condition $\sigma_0 = 1$.

Another sequence that needs to be evaluated is $\{\hat{p}_n^{(\lambda)}\}$, and also this task can be performed efficiently by exploiting coherence. We multiply both sides of (6.1) by $p_n^{(\lambda)}$ and evaluate the Sobolev inner product. Since $\langle p_{n+1}^{(\lambda)}, p_n^{(\lambda)} \rangle_\lambda = 0$, we have

$$\begin{aligned}\hat{p}_n^{(\lambda)} &= -\sigma_n \langle p_{n+1} - p_n, p_n^{(\lambda)} \rangle_\lambda \\ &= -\sigma_n \{ (p_{n+1}, p_n^{(\lambda)})_1 - (p_n, p_n^{(\lambda)})_1 + \lambda (q_n, p_n^{(\lambda)\prime})_2 \}\end{aligned}$$

and

$$(p_{n+1}, p_n^{(\lambda)})_1 = (q_n, p_n^{(\lambda)\prime})_2 = 0$$

gives

$$\hat{p}_n^{(\lambda)} = \sigma_n (p_n, p_n^{(\lambda)})_1. \tag{6.6}$$

Multiplying (6.1) by p_{n+1} and evaluating the $(\cdot, \cdot)_1$ inner product yield the identity

$$(p_{n+1}, p_{n+1}^{(\lambda)})_1 = \sigma_n \|p_{n+1}\|_1^2, \tag{6.7}$$

since orthogonality implies that $(p_{n+1}, p_n)_1 = (p_{n+1}, p_n^{(\lambda)})_1 = 0$. Shift of the index in (6.7) and substitution in (6.6) give

$$\hat{p}_n^{(\lambda)} = \sigma_{n-1}(\lambda) \|p_n\|^2. \tag{6.8}$$

To sum up, we managed to reduce the evaluation of Fourier–Sobolev coefficients to the evaluation of standard expansion coefficients with respect to $d\varphi$ and $d\psi$—a task that is easy and safe to accomplish by virtue of orthogonality—and a simple recursion.

ACKNOWLEDGMENTS

The authors thank Walter Gautschi (Perdue) and André Ronveaux (Namur), who kindly provided them with many references on orthogonality in Sobolev spaces, Richard Askey (Madison) and Lance Littlejohn (Utah) for their insightful remarks, and the anonymous referees for comprehensive and helpful reviews. In particular, one of the referee's reports has been instrumental in providing unifying treatment of classical Borel measures in Section 3. They also acknowledge many useful discussions with Brad Baxter and Martin Buhmann (Cambridge) and with Markus Hegland (Zurich).

REFERENCES

1. P. ALTHAMMER, Eine Erweiterung des Orthogonalitätsbegriffes bei Polynomen und deren Anwendung auf die beste Approximation, *J. Reine Angew. Math.* **211** (1962), 192–204.
2. R. ASKEY AND M. ISMAIL, Recurrence relations, continued fractions and orthogonal polynomials, *Mem. Amer. Math. Soc.*, **319** (1984).
3. H. BAVINCK AND H. G. MEIJER, Orthogonal polynomials with respect to a symmetric inner product involving derivatives, *Appl. Anal.* **33** (1989), 103–117.
4. J. BRENNER, "Über eine Erweiterung des Orthogonalitätsbegriffes bei Polynomen in einer und zwei Variablen," Ph.D. thesis, Techn. Hochschule Stuttgart, 1969.
5. J. BRENNER, Über eine Erweiterung des Orthogonalitätsbegriffes bei Polynomen, *in* "Constructive Theory of Functions," Akadémiai Kiadó, Budapest, 1972.
6. C. CANUTO AND A. QUARTERONI, Approximation results for orthogonal polynomials in Sobolev spaces, *Math. Comp.* **38** (1982), 67–88.
7. T. S. CHIHARA, "An Introduction to Orthogonal Polynomials," Gordon & Breach, New York, 1978.
8. E. A. COHEN, Zero distribution and behavior of orthogonal polynomials in the Sobolev space $W^{1,2}[-1, 1]$, *SIAM J. Math. Anal.* **6** (1975), 105–116.
9. W. GAUTSCHI, Minimal solutions of three-term recurrence relations and orthogonal polynomials, *Math. Comp.* **36** (1981), 547–554.
10. W. GRÖBNER, Orthogonale Polynomsysteme, die gleichzeitig mit $f(x)$ auch deren Ableitung $f'(x)$ approximieren, *in* "Funktionalanalysis, Approximationstheorie, Numerische Mathematik" (L. Collatz, G. Meinardus, and H. Unger, Eds.), Birkhäuser-Verlag, Basel, 1965.
11. W. HAHN, Über die Jacobischen Polynome und zwei verwandte Polynomklassen, *Math. Z.* **39** (1935), 634–638.
12. W. HAHN, Über Orthogonalpolynome mit besonderen Eigenschaften, *in* "E. B. Christoffel" (P. L. Butzer and F. Fehér, Eds.), Birkhäuser-Verlag, Basel, 1981.
13. W. HAHN, "Zur Theorie der Orthogonalpolynome," Forschungszentrum Graz, Bericht Nr. 190, 1983.
14. A. ISERLES, P. E. KOCH, S. P. NØRSETT, AND J. M. SANZ-SERNA, Orthogonality and approximation in a Sobolev space, *in* "Algorithms for Approximation" (J. C. Mason and M. G. Cox, Eds.), Chapman & Hall, London, 1990.
15. R. KUMAR, A class of quadrature formulas, *Math. Comp.* **28** (1974), 769–778.
16. R. KUMAR, Certain Gaussian quadratures, *J. IMA* **14** (1974), 175–182.
17. R. KOEKOEK, "Generalizations of Laguerre Polynomials," Univ. of Delft Tech. Rep. 89-51, 1989.
18. D. C. LEWIS, Polynomial least square approximations, *Amer. J. Math.* **69** (1947), 273–278.
19. L. L. LITTLEJOHN, W. N. EVERITT, AND S. C. WILLIAMS, Orthogonal polynomials in weighted Sobolev spaces, *in* "Orthogonal Polynomials and Their Applications" (J. Vinuesa, Ed.), Dekker, New York, 1989.
20. F. MARCELLAN AND A. RONVEAUX, "On a Class of Polynomials Orthogonal with Respect to a Discrete Sobolev Inner Product," Univ. of Namur Tech. Report, 1990.
21. P. MARONI, Sur la suite des polynômes orthogonaux associée à la forme $u = \delta c + \lambda(x-c)^{-1} L$, *in* "Actas V Simposium sobre Polinomios Ortogonales y Aplication Vigo" (A. Cachafeiro and E. Godoy, Eds.), 1988.
22. T. E. PRICE, JR., Orthogonal polynomials for nonclassical weight functions, *SIAM J. Numer. Anal.* **16** (1979), 999–1006.
23. E. D. RAINVILLE, "Special Functions," Macmillan, New York, 1960.